Numerical Solutions of Transient MHD Phenomena

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Abstract

A NUMERICAL model generating simultaneous solutions to the coulpled gasdynamic-electromagnetic equations of motion is developed. The transient relationship existing between the hydrodynamic flow properties and electromagnetic field quantities is explored. The parabolic gasdynamic equations of motion are discretized by a set of first-order accurate, explicit finite difference expressions and are solved in an Eulerian reference frame. Employing MHD approximations to Maxwell's relations, a linear second-order elliptic partial differential equation describing the current stream function ψ or the electric potential ϕ is developed and is solved by the method of successive overrelaxation. The time-dependent hydrodynamic and electromagnetic properties of a conducting fluid are evaluated by including the Lorentz force ($\bar{j}\times\bar{B}$) and the electrical energy ($\bar{j}\cdot\bar{E}$) in the equations of momentum and energy, respectively. Simultaneously, ϕ or ψ is computed as a function of the updated flow velocity, temperature, and pressure, demonstrating that stable simultaneous solutions with severe pressure gradients are obtainable with this technique.

Contents

An electrically conducting fluid flowing in the presence of a transverse magnetic field may either be accelerated or decelerated, depending on the orientation of the pondermotive or Lorentz force $\bar{j}\times\bar{B}$. If the flow is accelerated, as in an "accelerator," energy ($\bar{j}\cdot\bar{E}$) must be added to the fluid; whereas if the flow is decelerated, energy may be extracted from the fluid, as in an MHD generator. The governing equations describing the above phenomena consist of the fluid equations of motion: mass, momentum, energy, and equation of state, and the electromagnetic equations: Maxwell's relations and Ohm's law.

The development of a viable numerical procedure operating in the flow regime of MHD generators is hampered not only by typical numerical obstacles inherent in both gasdynamics and electromagnetic methods, but also by the unique difficulties arising from the marriage of these procedures. Because of these difficulties, various schemes[1-3] have been developed to solve the electromagnetic and gasdynamic equations for certain defined flowfields and configurations. In the present formulation, stable simultaneous solutions of the fluidic and electromagnetic equations are obtained. The governing fluidic equations are written as

$$(\mathrm{D}\rho/\mathrm{D}t)+\rho(\nabla\cdot\bar{v})=0 \tag{1}$$

$$\rho(\mathrm{D}\bar{v}/\mathrm{D}t)+\overline{\nabla}p+\nabla\cdot\bar{\bar{\tau}}-j\times\bar{B}+\rho\bar{g}=0 \tag{2}$$

$$\rho(\mathrm{D}/\mathrm{D}t)[e+(\bar{v}\cdot\bar{v}/2)]+\nabla\cdot(p\bar{v})-\nabla\cdot(\bar{\bar{\tau}}\cdot\bar{v})+\rho(\bar{v}\cdot\bar{g})$$
$$+\nabla\cdot\bar{q}-j\cdot\bar{E}=0 \tag{3}$$

$$p=p(\rho,T) \tag{4}$$

where $\rho,\bar{v},p,\bar{j},\bar{B},e$ and $\bar{E}$ are the density, velocity, pressure, current density, magnetic induction, total energy, and electric field, respectively; $\bar{\bar{\tau}}$ is the stress tensor, and $\bar{q}$ is the heat conduction. The electromagnetic equations are:

$$\nabla\times\bar{E}=-\partial\bar{B}/\partial t \tag{5}$$

$$\nabla\times\bar{B}=\mu_p j \tag{6}$$

$$j=\bar{\bar{\sigma}}\cdot(\bar{E}+\bar{v}\times\bar{B}) \tag{7}$$

$$\bar{\bar{\sigma}}=\frac{\sigma}{1+\beta^2}\begin{bmatrix}1 & -\beta\\ \beta & 1\end{bmatrix} \tag{8}$$

When the magnetic Reynolds number is sufficiently low, as occurs in the MHD generator, Eqs. (5-8) may be reduced to

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)+A\frac{\partial F}{\partial x}+B\frac{\partial F}{\partial y}+C=0 \tag{9}$$

In Eq. (9), F represents either the electric potential ϕ or the current stream function ψ; coefficients A, B, and C are functions of $\sigma,\beta,\bar{v},\bar{B}$, and their derivatives; and μ_p, σ, and β are the magnetic permeability, conductivity, and Hall parameter, respectively. The electric potential and current stream function are defined by

$$\bar{E}=-\overline{\nabla\phi},\ j_x=\partial\psi/\partial y,\ j_y=-\partial\psi/\partial x \tag{10}$$

The gasdynamic equations have been solved separately,[4] exclusive of electromagnetic effects, resulting in excellent agreement with known exact solutions and with experimental results. As with most finite difference formulations of the gasdynamics equations (alone), major difficulties arise in preserving numerical stability when computing in regions of adverse flow conditions (severe pressure gradients, etc.). To minimize this problem, a method is employed which treats the nonconservative form of Eqs. (1-4)[1] by time and space centering the nonconvective terms in the equations of motion; integrating in the Lagrangian reference frame; and finally transporting mass, momentum, and energy across cell boundaries by the "donor-receiver" technique. This method has clearly demonstrated that accurate solutions may be obtained in all flow regimes, subsonic through hypersonic, and in each case[4] was found to be free of any undesirable numerical instability.

Solutions to the electromagnetic equations for MHD flow, Eqs. (9) and (10), are considerably less difficult since the equations are linear and, therefore, expeditiously reducible to a set of algebraic expressions which in turn are easily solved. Instability can develop, however, for the problems characterized by either large Hall parameters or strongly varying electrical properties, e.g., conductivity.

The major problem for simultaneous solution of these coupled equations is in blending the two equation sets into an

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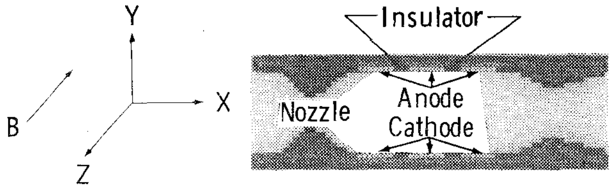

Fig. 1 MHD generator.

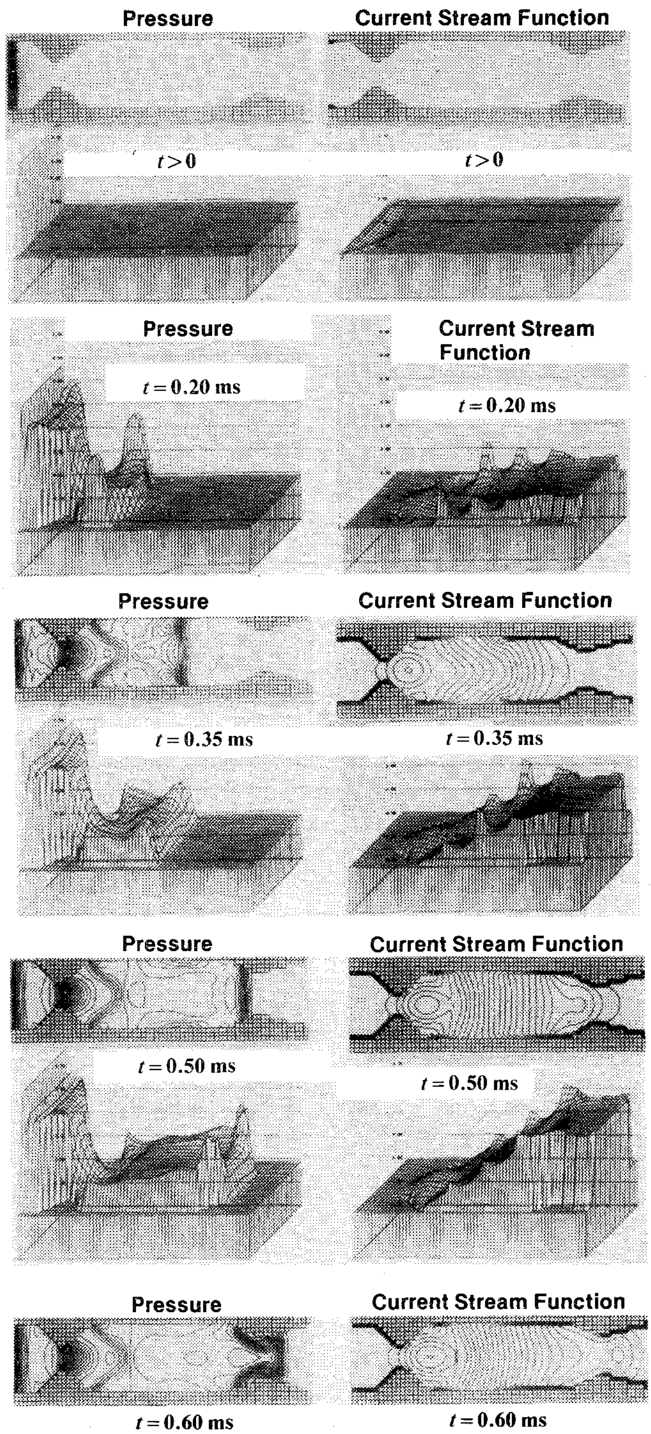

Fig. 2 Transient pressure and current stream function in an MHD generator.

autonomous, stable and converging integration scheme. Further, since time dependent calculations are, as a rule, computationally time consuming, it becomes essential that weakly coupled gasdynamic-electromagnetic regions be identified in order to avoid unnecessary computations. The present methodology for finding stable simultaneous solutions to the two sets of equations is unique and is presented in more detail in Ref. 4.

An MHD generator with three electrode pairs is shown in Fig. 1. The electrodes are seven cells wide and are separated by insulators of three cells thickness. Physically, each cell is 1 cm^2. Flow emanating from the chamber chokes in the nozzle and passes supersonically through the generator over the cathode-anode arrangement shown in Fig. 1. Electrical property values consist of a constant conductivity of 10 mhos/m, zero Hall current, and an externally applied magnetic field of 5 Teslas. Temperature and pressure of the flow in the generator were approximately 2500 K and 10 atm, respectively. It is assumed that the flow is inviscid and that both $\bar{g}$ and q are zero. Although these assumptions are important, they are not essential in demonstrating the stability or convergence of the present formulation.

The grid modeling the generator is 2-D planar and employs approximately 1500 cells. Results of the computations, Fig. 2, illustrate the time-dependent pressure and current stream function at four discrete time intervals during the transient operation. The pressure p and current stream function are plotted in the Z direction (opposing the orientation of the applied magnetic field). As the flow moves down the channel, the pressure and current stream function build up as noted in Fig. 2. At the second time step, $t=0.20$ ms, the three electrode current density spikes are clearly illustrated. In the last two time steps shown, $t=0.50$ ms and $t=0.60$ ms, note that the upstream throat and the downstream diffuser throat become choked and some shorting exists between the three electrode pairs. Lines of constant p and ψ are shown in the contour maps also displayed in Fig. 2.

Computation time, for 500 time integrations done on a CDC 6600, was approximately 30 min. Convergence of the electromagnetic quantities, Eqs. (9) and (10), required from 5 to 50 iterations per explicit "hydro" integration and thus consumes 50-80% of total cp time which doubles or triples the computational demands made by the same non-MHD gasdynamic application. Treating viscous flow further increases the computer time by 30 to 100%.

In conclusion, stable solutions for the MHD problem including severe pressure gradients (shock) have been demonstrated. The present methodology shows great promise for complex configurations if some numerical hurdles can be overcome; namely, that of resolving a sufficiently refined grid in order to preserve numerical accuracy and at the same time prevent computer run times from escalating beyond most research budgets.

References

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